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1990 J. Phys. A: Math. Gen. 23 537

(http://iopscience.iop.org/0305-4470/23/4/021)

J. Phys. A: Math. Gen. 23 (1990) 537-544. Printed in the UK

Anticommutator analogues of certain identities involving repeated commutators

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Received 28 June 1989

Abstract. The generalisation of the Baker-Hausdorff lemma and its anticommutator analogue is formulated. Additionally, the anticommutator analogues of several well known operator identities involving repeated commutators are derived. It is pointed out that these are more convenient for application whenever the operators in question satisfy simpler repeated anticommutator relations (and, in particular, when they anticommute) than the repeated commutator relations. Diagonalisation of two spin-1 Hamiltonians, in which the anticommutator analogue of the Baker-Hausdorff lemma is used to good advantage, is presented.

1. Introduction

Certain identities, like the Baker-Hausdorff lemma [1], (also known as the Lie series [2]), the Campbell-Baker-Hausdorff (CBH) formula [3] and similar have found diverse applications in classical [4-6] and quantum [7-11] physics. In the classical context these identities inolve the Poisson brackets†, while in quantum physics the commutators appear instead‡. This is, of course, because the identities in question are based only on the general algebraic properties which are the same for both the Poisson brackets and the quantum mechanical commutators (these being only two different realisations of the Lie product). Now, in quantum mechanics along with the commutators the anticommutators appear, in many cases, on an equal footing. These, however, have no classical analogues, and have different algebraic properties (in particular the anticommutators do not satisfy the Jacobi identity); therefore the above-mentioned identities do not apply to them. Having in mind the similar role played in quantum mechanics by the commutators and anticommutators in certain cases, it is legitimate to ask what analogous identities the anticommutators do satisfy (if any). This question has received little attention in the physics literature. Recently we showed [12] that the Baker-Hausdorff lemma has a closely related anticommutator analogue, and that this new identity can, in certain cases, be more convenient for application. In this paper we attempt a more systematic study of this problem and as a result derive the

† More precisely, they involve the differential operator

$$\hat{A}_-^{\text{classical}} \equiv \sum_k \left(\frac{\partial A}{\partial q_k} \frac{\partial}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial}{\partial q_k} \right)$$

acting on a phase space function $B = B(q, p)$ giving the Poisson bracket $\{A, B\}$.

‡ Again, more precisely, the commutator superoperator $\hat{A}_-$ appears, acting on an operator B giving the commutator $[A, B]$; see section 2 below.

anticommutator analogues of several operator identities involving repeated commutators. The treatment presented also shows that certain identities, like the Kubo identity and the CBH formula, do not possess direct anticommutator analogues.

In section 2 we summarise certain results pertaining to the repeated (anti)commutators. These are used to derive the anticommutator analogues of some well known identities involving repeated commutators. In section 3, exact diagonalisation of two spin-1 Hamiltonians, with the help of the anticommutator analogue of the Baker-Hausdorff lemma, is presented.

2. Theory

Firstly, we summarise some useful results pertaining to the repeated (anti)commutators. These we define by the successive application of the linear (anti)commutator superoperator $\hat{A}_{\pm}$ (a caret is used to denote a superoperator) as follows:

$$\hat{A}_{\pm}^{0}B \equiv B \tag{1}$$

$$\hat{A}_{\pm}^{1}B = \hat{A}_{\pm}B \equiv [A, B]_{\pm} \equiv AB \pm BA \tag{2}$$

$$\hat{A}_{\pm}^{n}B \equiv \hat{A}_{\pm}(\hat{A}_{\pm}^{n-1}B) \qquad n = 2, 3, \ldots . \tag{3}$$

Here, A and B denote two operators. Explicitly, for $n \geqslant 0$, we have [2]

$$\hat{A}_{\pm}^{n}B = \sum_{k=0}^{n} (\pm 1)^{k} \binom{n}{k} A^{n-k} B A^{k}. \tag{4}$$

Also of some interest is the relation expressing a repeated commutator in terms of repeated anticommutators and vice versa

$$\hat{A}_{\pm}^{n}B = \sum_{k=0}^{n} (\pm 2)^{k} \binom{n}{k} (\hat{A}_{\mp}^{n-k}B) A^{k}. \tag{5}$$

Both (4) and (5) are easily proved by induction.

Repeated (anti)commutators have a number of useful properties. To begin with, for $k \leqslant n$, we have trivially

$$\hat{A}_{\pm}^{n}B = \hat{A}_{\pm}^{k}(\hat{A}_{\pm}^{n-k}B). \tag{6}$$

Also

$$F\hat{A}_{\pm}^{n}B = \hat{A}_{\pm}^{n}(FB) \tag{7}$$

$$(\hat{A}_{\pm}^{n}B)F = \hat{A}_{\pm}^{n}(BF) = \hat{A}_{\pm}^{n}BF. \tag{8}$$

Here, $F = F(A)$ denotes a function of the operator A. Equation (8) shows, for example, that the last bracket in (5) can be dropped. Hereafter we shall use this property freely to simplify the notation. More generally, we have ($m = 1, 2, 3, \ldots$)

$$\hat{F}_{\sigma}^{m}(\hat{A}_{\pm}^{n}B) = \hat{A}_{\pm}^{n}(\hat{F}_{\sigma}^{m}B) = \hat{A}_{\pm}^{n}\hat{F}_{\sigma}^{m}B \tag{9}$$

with $\sigma = +$ or $\sigma = -$. The proofs of (7)-(9) are all based on (4).

Linearity

$$\hat{A}_{\pm}^{n}(\beta B + \gamma C + \ldots) = \beta\hat{A}_{\pm}^{n}B + \gamma\hat{A}_{\pm}^{n}C + \ldots \tag{10}$$

is also obvious from (4).

If an ordinary function is defined by the series expansion

$$f(x) \equiv \sum_n c_n x^n \tag{11}$$

then it is convenient to *define* a set ($k = 0, 1, 2, \ldots$) of the corresponding (anti)commutator superoperator functions via

$$^{(k)}f(\hat{A}_\pm) \equiv \sum_n c_n \hat{A}_\pm^{n+k-1}. \tag{12}$$

Here, terms with $n + k - 1 < 0$ (if any) are dropped by convention. The most important is the set of the exponential functions $^{(k)}\mathrm{e}^{\hat{A}_\pm}$. These functions are closely related

$$^{(1)}\mathrm{e}^{\hat{A}_\pm} = 1 + \hat{A}_\pm\, {}^{(0)}\mathrm{e}^{\hat{A}_\pm} \tag{13}$$

$$^{(k)}\mathrm{e}^{\hat{A}_\pm} = \hat{A}_\pm^{k-1}\, {}^{(1)}\mathrm{e}^{\hat{A}_\pm} = {}^{(1)}\mathrm{e}^{\hat{A}_\pm} \hat{A}_\pm^{k-1} \tag{14}$$

with $k = 2, 3, \ldots$. The validity of (14) is obvious from (12) and (6).

We have, more generally, for $k = 1, 2, \ldots$

$$^{(k)}\mathrm{e}^{\hat{A}_\pm} B = \hat{A}_\pm^{k-1}\, {}^{(1)}\mathrm{e}^{\hat{A}_\pm} B = {}^{(1)}\mathrm{e}^{\hat{A}_\pm} \hat{A}_\pm^{k-1} B = \hat{A}_\pm^{k-1}\, {}^{(1)}\mathrm{e}^{\hat{A}_\mp} B\, \mathrm{e}^{\pm 2A} \tag{15}$$

(where the first two equalities repeat (14), and the last equality is proved below). Also

$$^{(l)}\mathrm{e}^{\hat{A}_\mp} \hat{A}_\pm^{k-l} = \hat{A}_\mp^{l-1} \hat{A}_\pm^{k-l}\, {}^{(1)}\mathrm{e}^{\hat{A}_\mp} \tag{16}$$

which, for $l = 1$, reduces to

$$^{(1)}\mathrm{e}^{\hat{A}_\mp} \hat{A}_\pm^{k-1} = \hat{A}_\pm^{k-1}\, {}^{(1)}\mathrm{e}^{\hat{A}_\mp} \tag{17}$$

so that $^{(1)}\mathrm{e}^{\hat{A}_\mp}$ and $\hat{A}_\pm^{k-1}$ also commute (cf (14)). Equation (17) is to be compared with the last equality in (15).

Firstly, we prove the $k = 1$ case of (15). With the help of (12) and (5), introducing $m \equiv n - k$ and making the rearrangement for double summation, we get

$$\begin{aligned}
^{(1)}\mathrm{e}^{\hat{A}_\pm} B &= \sum_{n=0}^{\infty} \frac{\hat{A}_\pm^n}{n!} B = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\pm 2)^k}{k!(n-k)!} \hat{A}_\mp^{n-k} B A^k \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(\pm 2)^{n-m}}{(n-m)!m!} \hat{A}_\mp^m B A^{n-m} \\
&= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(\pm 2)^{n-m}}{(n-m)!m!} \hat{A}_\mp^m B A^{n-m} \\
&= \sum_{m=0}^{\infty} \frac{(\pm 2)^{-m}}{m!} \hat{A}_\mp^m B A^{-m} \left(\sum_{n=m}^{\infty} \frac{(\pm 2)^n}{(n-m)!} A^n \right).
\end{aligned}$$

Defining $l \equiv n - m$, the sum in the bracket gives $(\pm 2)^m A^m\, \mathrm{e}^{\pm 2A}$ and we get

$$^{(1)}\mathrm{e}^{\hat{A}_\pm} B = {}^{(1)}\mathrm{e}^{\hat{A}_\mp} B\, \mathrm{e}^{\pm 2A}$$

completing the proof of (15) for the $k = 1$ case. Applying $\hat{A}_\pm^{k-1}$ and using (14) we prove the validity of (15) for any k.

Next, we give a simple proof of (16) based on (14)

$$\begin{aligned}
^{(l)}\mathrm{e}^{\hat{A}_\mp}(\hat{A}_\pm^{k-l} B) &= \hat{A}_\mp^{l-1}\, {}^{(1)}\mathrm{e}^{\hat{A}_\mp}(\hat{A}_\pm^{k-l} B) \\
&= \hat{A}_\mp^{l-1} \hat{A}_\pm^{k-l}\, {}^{(1)}\mathrm{e}^{\hat{A}_\mp} B.
\end{aligned}$$

In the last step we used (9), which implies that $^{(1)}\mathrm{e}^{\hat{A}_\mp}$ and $\hat{A}_\pm^{k-l}$ commute.

Differentiation with respect to a continuous parameter α gives

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(\alpha\ {}^{(0)}\mathrm{e}^{\alpha\hat{A}_\mp}) = {}^{(1)}\mathrm{e}^{\alpha\hat{A}_\mp} \tag{18}$$

and $(k=0,1,2,\ldots)$

$$\frac{\mathrm{d}^k}{\mathrm{d}\alpha^k}\,{}^{(1)}\mathrm{e}^{\alpha\hat{A}_\mp} = \hat{A}_\pm^k\,{}^{(1)}\mathrm{e}^{\alpha\hat{A}_\mp} = \alpha^{-k}\,{}^{(k+1)}\mathrm{e}^{\alpha\hat{A}_\pm}. \tag{19}$$

In [12] we proved the Barker-Hausdorff lemma and its anticommutator analogue

$$\mathrm{e}^A B\,\mathrm{e}^{\pm A} = {}^{(1)}\mathrm{e}^{\hat{A}_\mp} B \tag{20}$$

or equivalently

$$\mathrm{e}^A B\,\mathrm{e}^{-A} = {}^{(1)}\mathrm{e}^{\hat{A}_-} B \tag{21a}$$

$$= {}^{(1)}\mathrm{e}^{\hat{A}_+} B\,\mathrm{e}^{-2A} \tag{21b}$$

by the differential equation method [3]. In fact, we see that the $k=1$ case of (19) was used in the proof. If one takes the Baker-Hausdorff lemma for granted, then we see that its anticommutator analogue follows directly from the $k=1$, lower-sign, case of (15). Anyway, we have more generally (see (19) and (20))

$$\frac{\mathrm{d}^k}{\mathrm{d}\alpha^k}(\mathrm{e}^{\alpha A} B\,\mathrm{e}^{\pm\alpha A}) = \alpha^{-k}\,{}^{(k+1)}\mathrm{e}^{\alpha\hat{A}_\mp} B \tag{22}$$

with $k=0,1,2,\ldots$.

If in (20) we change $B\to\mathrm{e}^B$ and then multiply from the right by $\mathrm{e}^{\mp A}$, we get

$$\mathrm{e}^A\,\mathrm{e}^B = {}^{(1)}\mathrm{e}^{\hat{A}_\mp}\,\mathrm{e}^B\,\mathrm{e}^{\mp A} \tag{23}$$

the lower-sign case of this operator identity being well known [5].

From (21) and the well known identities involving repeated commutators [5, 6] namely

$$({}^{(1)}\mathrm{e}^{\hat{A}_-} B)^n = {}^{(1)}\mathrm{e}^{\hat{A}_-} B^n \tag{24}$$

$${}^{(1)}\mathrm{e}^{\hat{C}_-} D = {}^{(1)}\mathrm{e}^{\hat{A}_-}\,{}^{(1)}\mathrm{e}^{\hat{B}_-} D \tag{25}$$

(in (25) and (28) $\mathrm{e}^C \equiv \mathrm{e}^A\mathrm{e}^B$, so that $C=\ln(\mathrm{e}^A\mathrm{e}^B)$)

$${}^{(1)}\mathrm{e}^{\hat{A}_-}\hat{B}_\pm C = [{}^{(1)}\mathrm{e}^{\hat{A}_-} B, {}^{(1)}\mathrm{e}^{\hat{A}_-} C]_\pm \tag{26}$$

etc, we get immediately the corresponding repeated anticommutator expressions

$$({}^{(1)}\mathrm{e}^{\hat{A}_+} B\,\mathrm{e}^{-2A})^n = {}^{(1)}\mathrm{e}^{\hat{A}_+} B^n\,\mathrm{e}^{-2A} \tag{27}$$

$${}^{(1)}\mathrm{e}^{\hat{C}_+} D\,\mathrm{e}^{-2C} = {}^{(1)}\mathrm{e}^{\hat{A}_+}({}^{(1)}\mathrm{e}^{\hat{B}_+} D\,\mathrm{e}^{-2B})\,\mathrm{e}^{-2A} \tag{28}$$

$${}^{(1)}\mathrm{e}^{\hat{A}_+}\hat{B}_\pm C\,\mathrm{e}^{-2A} = [{}^{(1)}\mathrm{e}^{\hat{A}_+} B\,\mathrm{e}^{-2A}, {}^{(1)}\mathrm{e}^{\hat{A}_+} C\,\mathrm{e}^{-2A}]_\pm \tag{29}$$

respectively. As in the case of the anticommutator analogue of the Baker-Hausdorff lemma, the last three identities are more convenient for application whenever the operators in question are such that the repeated anticommutators are simpler to evaluate than the corresponding repeated commutators.

Finally, we mention that the Kubo identity [13] follows from (19). Indeed, integrating this equation with respect to α, applying the result to an operator B and then multiplying by $e^{-\alpha A}$ from the left, we get

$$e^{-\alpha A}\,{}^{(1)}e^{\alpha\hat{A}_\mp}B = e^{-\alpha A}B + e^{-\alpha A}\int_0^\alpha \frac{d\alpha'}{\alpha'}\,{}^{(2)}e^{\alpha'\hat{A}_\pm}B.$$

Now (20), with $A \to \alpha A$, leads to

$$B\,e^{\pm\alpha A} - e^{-\alpha A}B = e^{-\alpha A}\int_0^\alpha \frac{d\alpha'}{\alpha'}\,{}^{(2)}e^{\alpha'\hat{A}_\pm}B.$$

The lower-sign case gives then the Kubo identity

$$\hat{B}_-\,e^{-\alpha A} = e^{-\alpha A}\int_0^\alpha \frac{d\alpha'}{\alpha'}\,{}^{(2)}e^{\alpha'\hat{A}_-}B. \tag{30}$$

From this derivation one can also see that there is no direct anticommutator analogue of this identity (writing ${}^{(2)}e^{\alpha'\hat{A}_-}B = \alpha'\hat{A}_-\,{}^{(1)}e^{\alpha'\hat{A}_-}B = \alpha'\hat{A}_-\,{}^{(1)}e^{\alpha'\hat{A}_+}B\,e^{-2\alpha' A} = \alpha'\,{}^{(1)}e^{\alpha'\hat{A}_+}(\hat{A}_-B)\,e^{-2\alpha' A}$ and inserting back into (30) is about the best one can do in this case).

To conclude, in this section, by treating the repeated commutators and anticommutators on equal footing, we obtained a unified treatment of some diverse operator identities, and additionally a number of new anticommutator analogues of identities involving repeated commutators (equations (20)-(23) and (27)-(29)). These analogues will be more convenient for application whenever the repeated anticommutators in question are simpler to evaluate than the corresponding commutators (and especially in the case when the operators in question anticommute).

3. Certain applications

In this section we present exact diagonalisation of two spin-1 Hamiltonians of physical interest in which the anticommutator analogue of the Baker-Hausdorff lemma (21*b*), is used to good advantage.

Firstly, we consider the reduced Hamiltonian of an asymmetric-top molecule, in a given vibrational state and in the rotational state $J = 1$, including sextic centrifugal distortion terms [14]

$$H = \alpha_1 + \beta_1 J_3^2 + (J_1^2 - J_2^2)(\alpha_2 + \beta_2 J_3^2) + (\alpha_2 + \beta_2 J_3^2)(J_1^2 - J_2^2) \tag{31}$$

with

$$\alpha_1 \equiv X + Y - 4\Delta_J + 8H_J \tag{32a}$$

$$\beta_1 \equiv Z - \tfrac{1}{2}(X + Y) - 2\Delta_{JK} + 4H_{JK} + 2H_{KJ} - \Delta_K + H_K \tag{32b}$$

$$\alpha_2 \equiv \tfrac{1}{4}(X - Y) - 2\delta_J + 4\eta_J \tag{32c}$$

$$\beta_2 \equiv 2\eta_{JK} - \delta_K + \eta_K. \tag{32d}$$

In (32) the coefficients are as follows: X, Y and Z are the effective principal rotational constants; Δ_J, Δ_{JK}, Δ_K, δ_J and δ_K are the quartic distortion coefficients; H_J, H_{JK}, H_{KJ}, H_K, η_J, η_{JK} and η_K are the sextic distortion coefficients.

The operators $J_i (i=1,2,3)$ satisfy the usual commutation relations

$$[J_l, J_m]_- = i\varepsilon_{lmn} J_n \tag{33}$$

valid for any J, and additionally

$$J_l J_m J_n + J_n J_m J_l = \delta_{lm} J_n + \delta_{nm} J_l \tag{34}$$

valid for spin 1 only [15]. In the present context, a useful set of algebraic relations arises when one defines the related set $K_i (i=1,2,3)$ of Hermitian spin-1 operators, namely

$$K_1 \pm iK_2 \equiv (J_1 \pm iJ_2)^2 = J_\pm^2 \qquad K_3 \equiv J_3 \tag{35}$$

from which, conversely,

$$J_1^2 = 1 - \tfrac{1}{2}(K_3^2 - K_1) \qquad J_2^2 = 1 - \tfrac{1}{2}(K_3^2 + K_1) \tag{36a}$$

$$J_1 J_2 = \tfrac{1}{2}(K_2 + iK_3) \qquad J_2 J_1 = (J_1 J_2)^\dagger = \tfrac{1}{2}(K_2 - iK_3). \tag{36b}$$

Using (34) one may show that the K_i operators satisfy the following algebraic relations:

$$[K_l, K_m]_+ = 2\delta_{lm} K_3^2 \qquad [K_l, K_m]_- = 2i\varepsilon_{lmn} K_n. \tag{37}$$

These combine to give

$$K_l K_m = \delta_{lm} K_3^2 + i\varepsilon_{lmn} K_n \tag{38}$$

in striking analogy with the 2×2 Pauli spin matrix algebra. In particular, (38) implies that $K_1^2 = K_2^2 = K_3^2 = J_3^2$, $K_1 K_2 = iK_3$ and similar.

Expressing (31) in terms of the new operators, we find

$$H = \alpha_1 + 2(\alpha_2 + \beta_2) K_1 + \beta_1 K_3^2. \tag{39}$$

In order to diagonalise this Hamiltonian, we define the unitary operator

$$U \equiv \exp(i\phi K_2/2) \tag{40}$$

so that

$$H' \equiv UHU^{-1} = \alpha_1 + 2(\alpha_2 + \beta_2) UK_1 U^{-1} + \beta_1 UK_3^2 U^{-1}. \tag{41}$$

Now, using the anticommutator analogue of the Baker–Hausdorff lemma (21b), and anticommutation property (37), we obtain at once

$$\begin{aligned} UK_1U^{-1} &= {}^{(1)}\exp(i\phi \hat{K}_{2,+}/2) K_1 \exp(-i\phi K_2) \\ &= K_1 \exp(-i\phi K_2) \\ &= K_1[1 + (\cos\phi - 1) K_3^2 - i \sin\phi K_2] \\ &= \cos\phi K_1 + \sin\phi K_3. \end{aligned} \tag{42}$$

Also, since K_3^2 commutes with K_2, one has $UK_3^2U^{-1} = K_3^2$, so that (41) becomes

$$H' = \alpha_1 + 2(\alpha_2 + \beta_2)(\cos\phi K_1 + \sin\phi K_3) + \beta_1 K_3^2. \tag{43}$$

The choice $\phi = \pi/2$ achieves, then, diagonalisation in the representation in which the 3×3 matrix $J_3 = \mathrm{diag}(1, 0, -1)$ ($\hbar = 1$).

Secondly, we consider the interaction Hamiltonian for a spin-1 nucleus, in a particular energy level, which includes electric quadrupole and magnetic dipole interactions [16]

$$H_{\mathrm{I}} = \frac{eQ}{4} \sum_{k,l=1}^{3} \left(\frac{\partial^2 V}{\partial x_k \partial x_l} \right)_{r=0} [J_k, J_l]_+ - \frac{eB}{2m} J_3. \tag{44}$$

Here, e is the charge, Q the size of the electric quadrupole moment and V is the electrostatic potential. The external electric field gradient tensor at the position of the nucleus is symmetric and traceless and may be chosen to be diagonal in principal axes [16], $(qe/2)\,\mathrm{diag}(\eta-1, -\eta-1, 2)$; the parameters q and η specify the size and orientation of the field gradient. Using this, and expressing J_i in terms of K_i operators, we obtain

$$H_I = \frac{\alpha}{\eta}(3K_3^2 - 2 + \eta K_1) - \beta K_3 \tag{45}$$

with

$$\alpha \equiv e^2 qQ/4 \qquad \beta \equiv eB/2m. \tag{46}$$

Using the same form of the unitary transformation (40), we get additionally (with the help of (21b) and (37), and in complete analogy with (42))

$$UK_3U^{-1} \stackrel{(1)}{=} \exp(\mathrm{i}\phi \hat{K}_{2,+}/2) K_3 \exp(-\mathrm{i}\phi K_2) = \cos\phi K_3 - \sin\phi K_1. \tag{47}$$

The transformed Hamiltonian $H_{\mathrm{I}}' = UH_{\mathrm{I}}U^{-1}$ becomes

$$H_{\mathrm{I}}' = \frac{\alpha}{\eta}(3K_3^2 - 2) + (\alpha\cos\phi + \beta\sin\phi)K_1 + (\alpha\sin\phi - \beta\cos\phi)K_3. \tag{48}$$

Choosing $\phi = \tan^{-1}(-\alpha/\beta)$, we obtain the diagonal form

$$H_{\mathrm{I}}' = \frac{\alpha}{\eta}(3K_3^2 - 2) \pm (\alpha^2 + \beta^2)^{1/2} K_3 \tag{49}$$

with the eigenvalues

$$\lambda_{1,3} = \frac{\alpha}{\eta} \pm (\alpha^2 + \beta^2)^{1/2} \qquad \lambda_2 = -2\alpha/\eta. \tag{50}$$

Thirdly, we briefly consider exact diagonalisation of the following general spin-1 Hamiltonian:

$$H = \alpha + \beta_1 K_1 + \beta_2 K_2 + \beta_3 K_3 + \gamma K_3^2. \tag{51}$$

Here $\alpha, \beta_1, \ldots, \gamma$ are restricted so that H is Hermitian. The Hamiltonian of this form implies that the original Hamiltonain is a linear combination of the following operators: 1, J_3, J_1^2, J_2^2, J_3^2, J^2, J_1J_2, J_2J_1, and possibly higher products of the form $J_1J_2J_3$, $J_3J_2J_1$ etc.

In this case one defines

$$N \equiv \beta_1 K_2 - \beta_2 K_1 \qquad U \equiv \exp(\mathrm{i}\phi N/2). \tag{52}$$

One has ($n = 1, 2, \ldots$)

$$N^{2n} = \beta^{2n} K_3^2 \qquad N^{2n+1} = \beta^{2n} N \tag{53a}$$

$$\hat{N}_+^n K_1 = -2^n (\beta_2/\beta^2) N^{n+1} \qquad \hat{N}_+^n K_2 = 2^n (\beta_1/\beta^2) N^{n+1} \tag{53b}$$

$$\hat{N}_+ K_3 = 0 \qquad \hat{N}_- K_3^2 = 0. \tag{53c}$$

Here $\beta \equiv (\beta_1^2+\beta_2^2)^{1/2}$. With the help of the anticommutator analogue of the Baker-Hausdorff lemma (21b), one obtains

$$UK_1U^{-1} = \cos(\beta\phi)K_1 + (\beta_1/\beta)\sin(\beta\phi)K_3 + (\beta_2/\beta^2)[1-\cos(\beta\phi)]N \tag{54a}$$

$$UK_2U^{-1} = \cos(\beta\phi)K_2 + (\beta_2/\beta)\sin(\beta\phi)K_3 + (\beta_1/\beta^2)[1-\cos(\beta\phi)]N \tag{54b}$$

$$UK_3U^{-1} = -\frac{\sin(\beta\phi)}{\beta}(\beta_1K_1+\beta_2K_2) + \cos(\beta\phi)K_3 \tag{54c}$$

$$UK_3^2U^{-1} = K_3^2. \tag{54d}$$

The transformed Hamiltonian is therefore

$$H' = \alpha + \beta_1[\cos(\beta\phi) - (\beta_3/\beta)\sin(\beta\phi)]K_1 + \beta_2[\cos(\beta\phi) - (\beta_3/\beta)\sin(\beta\phi)]K_2 + (\beta\sin(\beta\phi) + \beta_3\cos(\beta\phi))K_3 + \gamma K_3^2. \tag{55}$$

Choosing $\phi = \beta^{-1}\tan^{-1}(\beta/\beta_3)$, one achieves diagonalisation

$$H' = \alpha + (\beta_1^2+\beta_2^2+\beta_3^2)^{1/2}K_3 + \gamma K_3^2 \tag{56}$$

with the corresponding eigenvalues

$$\lambda_{1,3} = \alpha + \gamma \pm (\beta_1^2+\beta_2^2+\beta_3^2)^{1/2} \qquad \lambda_2 = \alpha. \tag{57}$$

Thus, the anticommutator analogue of the Baker-Hausdorff lemma provides an alternative, sometimes more convenient and efficient way of performing similarity or unitary transformation. This is especially true when the operators in question (as is the case in the above examples) have the anticommutation property.

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